Photometric Functions for Photoclinometry and Other Applications

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Received April 17, 1987, revised April 22, 1991

Least-squared fits to the brightness profiles across a disk or "limb darkening" described by Hapke's photometric function are found for the simpler Minnaert and lunar-Lambert functions. The simpler functions are needed to reduce the number of unknown parameters in photoclinometry, especially to distinguish the brightness variations of the surface materials from that due to the resolved topography. The limb darkening varies with the Hapke parameters for macroscopic roughness (\(\theta\)), the single-scattering albedo (\(w\)), and the asymmetry factor of the particle phase function (\(g\)). Both of the simpler functions generally provide good matches to the limb darkening described by Hapke's function, but the lunar-Lambert function is superior when viewing angles are high and when \(\theta\) is less than 30°. Although a nonunique solution for the Minnaert function at high phase angles has been described for smooth surfaces, the discrepancy decreases with increasing \(\theta\) and virtually disappears when \(\theta\) reaches 30° to 40°. The variation in limb darkening with \(w\) and \(g\), pronounced for smooth surfaces, is reduced or eliminated when the Hapke parameters are in the range typical of most planetary surfaces; this result simplifies the problem of photoclinometry across terrains with variable surface materials. The Minnaert or lunar-Lambert fits to published Hapke models will give photoclinometric solutions that are very similar (< 1° slope discrepancy) to the Hapke-function solutions for nearly all of the bodies and terrains thus far modeled by Hapke's function.

INTRODUCTION

High-resolution topographic information has applications to many planetary problems. The measurement of topography from photometric shading is known as "photoclinometry" by planetary scientists and as "shape-from-shading" within the computer vision community. Although topography can be derived by stereophotogrammetry, radar altimetry, and other methods, photoclinometry from spacecraft images has several significant advantages: (1) it can provide very high resolution measurements, limited only by the picture resolution; (2) it is rapid and relatively inexpensive; and (3) the method is particularly accurate for measuring slopes, rather than altitudes.

Radiometrically calibrated spacecraft images measure the brightness of a scene under specific angles of illumination (\(i\)), emission (\(e\)), and phase (\(\alpha\)). For an object without an optically significant atmosphere, this brightness is controlled by two basic classes of information: (1) the intrinsic properties of the surface materials, including composition, grain size, roughness, and porosity; and (2) variations in brightness due to the topography of the surface. Brightness variations controlled by (1) will be referred to as the "intrinsic albedo" in this paper. Many publications have focused on the first class of information (see references). Consideration of which photometric equations and input parameters are most appropriate for photoclinometry is the subject of this paper.

Recent theoretical work has been directed toward the derivation of semiempirical photometric functions from physical laws (Goguen 1981, Hapke 1981, 1984, 1986, Lumme and Bowell 1981). Among these functions, the equations of Hapke have been the most widely used and have proven accurate in laboratory and planetary measurements (e.g., Clark and Roush 1984, Veverka et al. 1986). The Hapke parameters (except \(g\)) are generally believed to be more easily related to physical properties of planetary surface materials than are the empirical parameters of other photometric functions. As a result, Hapke-function photometric models have become popular (see references). The purpose of this paper is to provide a means of relating these Hapke models to photometric functions more amenable to photoclinometry. Three Hapke input parameters are significant at phase angles greater than about 5°: the single-scattering albedo (\(w\)), the asymmetry factor of the particle phase function (\(g\)), and the average slope angle describing the subresolution macroscopic roughness (\(\theta\)).

The Hapke function is a complicated expression, especially when the effects of macroscopic roughness are included (Hapke 1984). Simpler photometric functions are faster and more convenient for use in photoclinometry. Also, as shown in this paper, the deviations between Hapke's function and the simpler functions are usually insignificant, i.e., less than the scatter in most planetary
of the photometric function (or “limb-darkening”) and observations, except near the limb or terminator. Furthermore, slope calculations result from the $\epsilon$, $\phi$-dependence of the photometric function (or “limb-darkening”) and whether or not Hapke’s function actually describes the $\epsilon$, $\phi$-dependence of planetary surfaces more accurately than the simpler functions has only been tested in two cases: (1) McEwen (1990) showed that a modified form of Hapke’s equation (but not the original function) best describes the $\epsilon$, $\phi$-dependence of Triton; and (2) McEwen et al. (1991) found that the limb-darkening of Io at low phase angles was more accurately described by the Minnaert function than by the Hapke function.

Perhaps the major problem in photoclinometry is how to separate brightness variations due to the intrinsicbedo of the surface materials from brightness variations due to topography. Most previous workers in photoclinometry have simply interpreted all of the brightness changes as due to topography, but this may result in large slope errors unless the surface materials have uniform photometric properties or the solar incidence angles are large. Davis and Soderblom (1984) distinguished the intrinsic reflectivity from topography by measuring two profiles of symmetrical impact craters from single images, assuming that each profile was identical in intrinsic albedo and slope, and by determining ratios of the Minnaert-function values along each profile to “cancel out” the surface reflectivities. A similar method, but one that utilizes two images of the same area rather than two profiles from the same image, has been called “two-image photoclinometry” (McEwen 1985). Use of three images to determine the two components of surface orientation and intrinsic albedo has been discussed by Woodham (1980), although the two-dimensional photoclinometry of Kirk (1986) allows reliable solution for the two components of surface orientation from a single image. The latter methods are difficult to employ with the photometric function of Hapke because there are more than two unknown parameters. Herkenhoff and Murray (1990) developed a two-image photoclinometric method with Hapke’s function by using an iterative solution for $w$ and slope and assuming that $g$ and $\theta$ did not vary along the profile. A more general photoclinometry method using Hapke’s equation would require four images in order to solve for slope, $w$, $g$, and $\theta$ at each position along a profile, but four images of an area acquired through the same spectral filter and with comparable resolutions do not usually exist.

The contribution of this paper is straightforward. Variations in brightness as a function of $\epsilon$ and $\phi$ (or the cosines of these angles, $\mu$ and $\mu_0$, respectively) are described by Hapke’s function with various input parameters. Best-fit approximations to the Hapke-function brightness variations are found for the Minnaert and lunar-Lambert functions. These functions each have one “limb-darkening” parameter that varies with phase angle ($\alpha$). The results are presented as plots of the limb-darkening parameters versus $\alpha$ for each set of Hapke input parameters. A preprint of this paper has been used by Schenk (1989, 1991) for photoclinometry of craters on icy satellites of Saturn and Uranus.

In addition to providing photometric functions for photoclinometry, the work presented here has several other practical applications: (1) The empirical functions are easier to use than Hapke’s function for computing normal albedos and for “flattening” spacecraft images prior to mosaicking. (2) The limb darkening of a body may be described in terms of the simpler functions, and then related to Hapke’s function (cf. Simonelli and Veverka 1986, Helfenstein and Veverka 1987, Helfenstein et al. 1988, McEwen et al. 1988, Veverka et al. 1989). (3) Many previous results for the Minnaert or lunar-Lambert photometric properties of a body only apply to specific phase angles. The results presented here provide a reasonable means of predicting the photometric behavior at other phase angles. (4) The relationships described in this paper may be used to compare the photometric results of different workers for various planetary bodies and laboratory materials.

Hapke model results for planets and satellites without optically significant atmospheres are summarized in Table I. Mars is excluded from Table I because the atmosphere, although optically thin during much of the year, nevertheless has a significant effect on the $\epsilon$, $\phi$-dependence. The values of $w$ and $g$ listed in Table I are those appropriate at wavelengths of about 0.5 $\mu$m, such as the Voyager clear (0.47 $\mu$m) or blue (0.48 $\mu$m) filters.

2. PHOTOMETRIC FUNCTIONS

A photometric function that describes the scattering from dark porous materials such as the regolith of our Moon is of the form

$$I(\mu, \mu_0, \alpha) = f(\alpha)\mu_0/(\mu + \mu_0), \quad (1)$$

where $I(\mu, \mu_0, \alpha)$ is the reflectance function, defined as the intensity of scattered light with the geometry indicated by $\mu$, $\mu_0$, and $\alpha$ divided by the incident solar flux times $\pi$; $f(\alpha)$ is a function of phase angle. $I(1, 1, 0)$ is normal albedo, $r_\alpha$. This function closely approximates the scattering behavior of many low-albedo particulate materials with macroscopically smooth surfaces measured in the laboratory (Veverka et al. 1978). However, Eq. (1) only precisely describes the scattering of the Moon at $\alpha = 0$, due to the effects of macroscopic roughness (Hapke 1966).

Scattering functions for particulate surfaces of any albedo have been derived from radiative transfer laws by Goguen (1981), Hapke (1981), and Lumme and Bowell.
TABLE 1
Hapke Photometric Parameters for Planetary Bodies

<table>
<thead>
<tr>
<th>Planet or satellite</th>
<th>$\omega^a$</th>
<th>$\varphi^a$</th>
<th>$\bar{\theta}$ (°)</th>
<th>M</th>
<th>L.L.</th>
<th>Source</th>
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<td>Moon</td>
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<td>10</td>
<td>7</td>
<td>6</td>
<td>Hillier et al. (1990)</td>
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</table>

* Values of $\omega$ and $\varphi$ appropriate at $\sim 0.5 \mu m$.

* rms deviation $\times 10^3/r_s$ (see text). M indicates Minnaert function; L.L indicates lunar-Lambert function.

* Used two-lobed Henyey-Greenstein function; $D_{rm}$ not evaluated.
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1.2

1.0

0.7

0.6

0.0 20 40 60 80 100 120 140 180

\( a \) (degrees)

FIG. 1. Plots of Minnaert \( k(a) \) exponents best fit to Hapke's function with \( \bar{\theta} = 0^\circ \) and \( g = -0.1 \) along photometric equator (solid lines) and mirror meridian (dashed lines). Also plotted are measured values for MgO-charcoal mixture (from Goguen 1981) with normal albedo of 0.28 along photometric equator (squares) and mirror meridian (solid circles).

1981). Hapke's function (without macroscopic roughness) can be expressed as

\[
I(\mu, \mu_0, \alpha) = \frac{1}{4\mu_0} I_0(\mu_0, \mu_0) S(\alpha, \mu_0) P(\alpha, g) + H(w, \mu) H(w, \mu_0) - 1, \tag{2}
\]

where \( S(\alpha, h) \) is the shadowing function (Hapke 1986), \( h \) is the compaction parameter, \( P(\alpha, g) \) is the single-particle phase function, \( g \) is the asymmetry parameter for the Henyey–Greenstein phase function (Henyey and Greenstein 1941), and \( H(w, \mu) \) and \( H(w, \mu_0) \) are Hapke's approximations to the functions tabulated by Chandrasekhar (1950). (See Hapke (1981, 1986) and Veverka et al. (1986) for further elaboration.)

Equation (2) has proven accurate for describing the scattering of smooth particulate laboratory samples of arbitrary albedo (Hapke and Wells 1981; Johnson et al. 1983; Gradie and Veverka 1984). However, planetary surfaces are not smooth like laboratory samples, but are rough on scales ranging from that of clumps of grains to large topographic features (Hapke 1984). Most of the macroscopic roughness is probably dominated by small-scale roughness because cohesive forces and material strengths are size dependent (Hapke 1984), but roughnesses at larger scales may be significant (Helfenstein 1988). The reflectance function with macroscopic roughness (Hapke 1984, p. 51–53) is very long and will not be repeated here.

There is a problem with Hapke's roughness model for very high albedo surfaces, at least as a function of phase angle (Buratti and Veverka 1985), because Hapke's model does not include the partial illumination of shadows by multiple scattering. In addition, Helfenstein (1988) reported that Hapke's equation does not accurately describe the scattering behavior of surfaces with \( \bar{\theta} \) greater than about 10° at large incidence and phase angles. These prob-
lems will not affect photoclinometric solutions provided Hapke's equation fits the actual $\mathbf{\epsilon}$-, $\mathbf{e}$-dependence.

Minnaert's function (Minnaert 1941) is a widely used empirical function, given by

$$I(\mu, \mu_0, \alpha) = B_0(\alpha)\mu^k(\mu^1 - 1), \quad (3)$$

where $B_0(\alpha) = I(1, 1, \alpha)$, and $k(\alpha)$ is an empirical exponent that varies with $\alpha$. Because of the laboratory and theoretical work of Goguen (1981), Minnaert's function has been regarded as inadequate except when $\alpha$ is near 0°. Veverka et al. (1986, p. 371) state that

"It is well known that Eq. (8) [Eq. (3) of this paper] is only a crude approximation to the scattering properties of real surfaces (see, e.g., Goguen, 1981), and that its indiscriminate application can lead a novice photometrist to confusion and specious results. The problem is that when Eq. (8) [Eq. (3)] is fitted to the scattering properties of real surfaces at phase angles in excess of about 30°, the value of $k$ derived is not unique, but depends on what subset of points on the planet's disk is used."

Goguen (1981) showed that his function, which is similar to the equation of Hapke (1981), does not exhibit this problem of uniqueness. However, by fitting Minnaert's function to Hapke's function, I will demonstrate that (1) the uniqueness problem with the Minnaert function greatly diminishes when the Hapke parameters $\bar{\theta}$ and $g$ are in the range typical of most planetary bodies, and (2) even for smooth surfaces, the problem is overstated by Veverka et al. (1986) because as $\alpha$ increases the dependence of the reflectance function on $k$ decreases.

Another empirical photometric function which has been widely utilized in recent years is a combination of the lunarlike function (Eq. (1)) and Lambert's function ($I(\mu_0, \alpha)$ proportional to $\mu_\alpha$). This "lunar-Lambert" function can be expressed in several forms. One form is

$$I(\mu, \mu_0, \alpha) = A f(\alpha)\mu_0/(\mu + \mu_0) + B \mu_0 \quad (4)$$

where $A$ and $B$ are empirical parameters which may or may not vary with $\alpha$. In the form of Eq. (4), the lunar-Lambert function is difficult to use for photoclinometry with either the symmetric method (Davis and Soderblom 1984) or the two-image method (McEwen 1985). The problem is that measurements from two profiles, i.e., two versions of Eq. (4), cannot be ratioed to cancel out effects of the intrinsic reflectivity of the surface materials. The lunar-Lambert function can be used for two-profile pho-
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3. FITS TO HAPKE'S FUNCTION

Best-fit values to Hapke's function for $k$ in Eq. (3) and for $L$ in Eq. (5) were computed by the following procedure. Sets of albedo values were computed along the photometric equator and along the mirror meridian by

$$I(\mu, \mu_0, \alpha) = B_0(\alpha)[2L(\alpha)\mu_0/(\mu + \mu_0) + (1 - L(\alpha))\mu_0],$$

where $L(\alpha) = A_f(\alpha)/(A_f(\alpha) + 2B)$. The distinction between Eqs. (4) and (5) are that the $\nu$, $e$-dependence in Eq. (5) is assumed independent of the intrinsic albedo (expressed as $B_0(\alpha)$).

FIG. 5. Comparison of Hapke function and best-fit Minnaert function with $\theta = 0^\circ$, $w = 0.1$, and $g = -0.1$, as in Fig. 2, but using $k(\alpha)$ values fit to both photometric equator and mirror meridian.

FIG. 6. Same as Fig. 5, but with $w = 0.99$. 
TABLE IIA
Minnaert Function Rms Deviation \( \times 10^3/r_n \)
\[
\begin{array}{ccccccc}
\hline
\theta & w: & 0.1 & 0.7 & 0.9 & 0.99 & 0.1 & 0.7 & 0.9 & 0.99 \\
\hline
0^\circ & 26 & 23 & 19 & 15 & 5 & 7 & 7 & 8 \\
10^\circ & 17 & 15 & 13 & 10 & 5 & 5 & 6 & 6 \\
20^\circ & 9 & 9 & 9 & 11 & 4 & 3 & 4 & 6 \\
30^\circ & 5 & 7 & 10 & 16 & 2 & 3 & 4 & 7 \\
40^\circ & 6 & 9 & 12 & 17 & 2 & 3 & 5 & 8 \\
50^\circ & 9 & 13 & 17 & 23 & 4 & 6 & 7 & 10 \\
\hline
\end{array}
\]

TABLE IIB
Lunar-Lambert Rms Deviations \( \times 10^3/r_n \)
\[
\begin{array}{ccccccc}
\hline
\theta & w: & 0.1 & 0.7 & 0.9 & 0.99 & 0.1 & 0.7 & 0.9 & 0.99 \\
\hline
0^\circ & 1 & 5 & 7 & 9 & 0 & 2 & 3 & 5 \\
10^\circ & 8 & 9 & 8 & 9 & 2 & 3 & 3 & 4 \\
20^\circ & 6 & 6 & 6 & 8 & 2 & 3 & 3 & 4 \\
30^\circ & 5 & 7 & 10 & 15 & 2 & 3 & 4 & 7 \\
40^\circ & 7 & 11 & 15 & 20 & 3 & 4 & 6 & 9 \\
50^\circ & 11 & 16 & 20 & 27 & 5 & 7 & 9 & 12 \\
\hline
\end{array}
\]

using Hapke's equation with various input parameters. (The photometric equation is the great circle passing through the subsolar and spacecraft points, and the mirror meridian is the perpendicular great circle where \( \mu = \mu_0 \); see Harris 1961.) Photometric longitude, \( \omega \), is measured along the photometric equator, and photometric latitude, \( \psi \), is measured along the mirror meridian. Points were evenly spaced in terms of \( \sin \omega \) and \( \sin \psi \). The \( \epsilon, \delta \)-dependence in Hapke's function is controlled almost entirely by the parameters \( w \), \( g \), and \( \theta \). The shadowing function has a minor effect except when \( \alpha \) is less than about 5°, but photoclinometry is unreliable at low phase angles (Davis and McEwen 1984). Therefore, \( h \) was set to 0.05 and the magnitude of the opposition effect was set to 1, values typical of many planetary surfaces (Hapke 1986). From Table 1, we see that published models for \( \theta \) vary from 3° to 36° and \( g \) varies from \(-0.06\) to \(-0.4\). Therefore, I have computed Hapke-function profiles with \( \theta \) varying from 0° to 50° in intervals of 10° and with \( g \) equal to 0.0 or \(-0.4\), to bracket the published model values. Next, the values for \( k \) and \( L \) resulting in the smallest root-mean-square (rms) deviation between the reflectance function from Eqs. (3) and (5) and the Hapke function were found. The Minnaert function predicts values near the limb approaching either infinity (when \( k \) is less than one) or zero (when \( k \) is greater than one). The lunar-Lambert function is also sometimes difficult to fit to Hapke's function near the limb. Therefore, values near the planetary limb (emission angle greater than 70° + \( \alpha/9 \)) were excluded from the rms calculation. Such oblique viewing angles are not desirable for photoclinometry and the poor fits at high emission angles will only occasionally be a problem.

3.1 Fits with Minnaert Function

The Minnaert function has been widely used for photoclinometry and for other applications. Two major problems with this function are (1) the divergence of \( I(\mu, \mu_0, \alpha, \epsilon, \delta) \),
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FIG. 8. Comparison of Hapke (solid lines) and Minnaert (dashed lines) functions with $\theta = 30^\circ$, $w = 0.99$, and $g = -0.1$ along photometric equator (a) and mirror meridian (b).

$\alpha$) toward infinity or zero as $e$ approaches $90^\circ$ and (2) the reported nonunique solutions for $k$ as a function of position on the disk. Because of the first problem the Minnaert function cannot be used with confidence at high emission angles. The second problem is analyzed in detail below.

Best-fit values for $k$ as a function of $\alpha$ along the photometric equator and mirror meridian, with $\theta = 0$ and $g = -0.1$, are shown in Fig. 1. The values for $k$ in each direction agree only near zero degrees phase. This discrepancy was first demonstrated by Goguen (1981). Some of Goguen's laboratory measurements are plotted on Fig. 1; the divergence of $k$ with increasing $\alpha$ is similar to that from the Hapke-function fits. The best-fit $k$ values fit Hapke's equation very closely, except near the limb, when different $k$ values are used along the $\omega$ and $\psi$ directions (Figs. 2 and 3).

Least-squared fits to $k(\alpha)$ along both the $\omega$ and $\psi$ directions (Fig. 4) are closer to the $k(\alpha)$ fits along the mirror meridian than along the photometric equator. This differ-

FIG. 9. Best-fit $k(\alpha)$ with $\theta = 10^\circ$.

FIG. 10. Best-fit $k(\alpha)$ with $\theta = 20^\circ$. 
ence is explained by the decreasing dependence of $I(\mu, \mu_0, \alpha)$ on $k$ along the photometric equator at high $\alpha$; therefore, a wide range of $k$ values will provide satisfactory fits. The same is true for $L$ in the lunar-Lambert function. Therefore, the discrepancy of $k$ values as a function of $\omega, \psi$ profile directions is not as severe as might be thought. Fits to Hapke's function with best-fit $k(\alpha)$ values along both the $\omega$ and the $\psi$ directions are shown in Figs. 5 and 6. The fits at high phase angles are poorer than those shown in Figs. 2 and 3, but may be adequate for some purposes. For example, the fits in Figs. 5 and 6 are better than the fits of Hapke's function to some planetary observations. However, the lunar-Lambert function gives a better fit (see below), and is the superior function to use when $\bar{\theta}$ is low and $\alpha$ is high.

3.2 Measurement of Deviations

Rather than finding different $k$ values as a function of $\omega$ or $\psi$, a better measure of the fit is the rms deviation of the empirical reflectance function versus Hapke's function. The deviation measure used in this paper is $D_{\text{rms}}$, the rms deviation divided by normal albedo ($r_n$) and multiplied by $10^3$. Shown in Table II are the mean $D_{\text{rms}}$ values (mean for all phase angles) for best-fit $k$ and $L$ values, fit as a function of $\omega$ and $\psi$ together. As examples, the $D_{\text{rms}}$ values in Fig. 6 are 8 (at $\alpha = 0^\circ$), 18 (at $\alpha = 60^\circ$), and 17 (at $\alpha = 120^\circ$). $D_{\text{rms}}$ is computed for points with $e$ less than $70^\circ + \alpha/9$, as discussed previously. From Table II it is clear that the lunar-Lambert function provides a better match to Hapke's function than the Minnaert function when $\bar{\theta}$ is
less than 30° and that both functions provide a closer match when \( g \) is set to -0.4.

Also shown in Table I are the \( D_{\text{rms}} \) values corresponding to the specific Hapke parameters derived for each planetary body or terrain. Because each of these bodies or terrains is characterized by either a high \( \theta \) (>20°) or a low \( g \) (< -0.1) or both, the \( D_{\text{rms}} \) values are low (<10) for every entry in Table I except for one Minnaert-function fit.

### 3.3 Errors in Photoclinometry

It is difficult to specify what \( D_{\text{rms}} \) values are acceptable for photoclinometry (assuming that the Hapke function fits the surface) because the errors in photoclinometry are a complicated function of \( \iota, \epsilon, \alpha \), intrinsic albedo variations, geometric control, and photometric model (Howard et al. 1982, Davis and McEwen 1984, Kirk 1986, Moore et al. 1986, Jankowski and Squyres 1990). Any particular photometric model may or may not give “acceptable” results (for example, ±1° maximum slope errors) depending on the other factors. There are cases in which no photometric function will give an accurate solution and there are cases where almost any function gives a well-constrained solution.

If we assume that the intrinsic albedo is uniform, that the geometric control is perfect, that slope azimuths are known, and that the Hapke function describes the actual scattering behavior, then the potential slope errors may be estimated from calculated disk scans such as those in Figs. 2, 3, 5, 6, 8, and 15. Slope errors are given by the difference in angular position of \( \psi \) or \( \omega \) corresponding to a particular reflectance from each reflectance function. For example, from Fig. 5a, \( I(\mu, \mu_0, 60°) \) of 0.004 is given at \( \omega \) values of 19.3° by the Hapke function and 22.3° by the Minnaert function, so the slope discrepancy is 3°. \( D_{\text{rms}} \) in this example is 23. If \( I(\mu, \mu_0, \alpha) \) versus \( \psi \) or \( \omega \) is flat, then there is no photoclinometric solution. From Fig. 5 we can see that the solution will be very poor for dark surfaces near 0° phase angle or when surface facets are tilted in the direction of the mirror meridian. The presence

![FIG. 15. Comparison of lunar-Lambert and Hapke functions with \( \bar{\theta} = 0° \), \( w = 0.99 \), and \( g = -0.1 \) along photometric equator (a) and mirror meridian (b).](image1)

![FIG. 16. Best-fit \( L(\alpha) \) with \( \bar{\theta} = 10° \).](image2)
of macroscopic roughness only slightly improves this situation. High-\(w\) surfaces result in stronger limb-darkening and stronger photoclinometric solutions at low phase angles and when surface facets are tilted in the direction of the mirror meridian, but the solution remains weak near the subsolar point (Fig. 6). Both bright and dark surfaces tilted in the direction of the photometric equation give strong photoclinometric solutions at high phase angles near the terminator, even if the photometric function only crudely approximates the surface photometry, because of the steep slope of \(I(\mu, \mu_0, \alpha)\) versus \(\psi\).

Excluding high-\(e\) observations and those conditions described above in which photoclinometry is unreliable, then those fits to Hapke's function with \(D_{\text{rms}}\) less than 10 will result in maximum slope errors of 1° or less. All of the lunar–Lambert and all but one of the Minnaert fits listed in Table 1 have \(D_{\text{rms}}\) of less than 10. This result indicates that the Minnaert or lunar–Lambert functions will give photoclinometric solutions that are very similar (<1° discrepancy) to Hapke-function solutions for nearly all of the bodies and terrains thus far modeled by Hapke's function.
3.4 Effects of Macroscopic Roughness

The discrepancy between $k$ values along the photometric equator and mirror meridian is greatly reduced with increasing macroscopic roughness. With $\theta$ set to 10°, $k(\alpha)$ is much more consistent (Fig. 7a); with $\theta$ set to 30°, the discrepancy in $k(\alpha)$ does not exceed 0.2 with $\alpha$ less than 90° (Fig. 7b). (With $\alpha > 90°$, the reflectance function has only a weak dependence on $k$.) $D_{rms}$ values are lowest when $\theta$ equals 20–30° (Table II), values typical of most planetary surfaces (Table I). Therefore, the effect of macroscopic roughness explains why the Minnaert function works poorly on smooth laboratory samples, but works well on many real planetary bodies, even at high phase angles.

Another effect of Hapke’s macroscopic roughness model is to increase the complexity of the $\epsilon$, $\epsilon$-dependence, especially at high values of $\omega$ (Fig. 8). As a result, the Minnaert and lunar–Lambert functions cannot fit the Hapke function precisely. This problem becomes severe when $\theta$ reaches 50°, but it is unlikely that many planetary surfaces are so rough. For values of $\theta$ from 30° to 40° the problem is significant ($D_{rms} > 10$) only for very bright surfaces ($\omega > 0.9$—see Table I), but there is already uncertainty concerning Hapke’s roughness model for bright surfaces (Buratti and Veverka 1985). Planetary observations matching the complex Hapke profiles shown in Fig. 8 have not been described, so the Minnaert or lunar–Lambert profiles may be better models for actual planetary surfaces that are both very rough and very bright.

The value of $\theta$ used with photoclinometry must represent only roughness at a scale smaller than the picture resolution, and should not include the actual topography being measured. Much of the roughness probably occurs on the scale of millimeters to meters (Hapke 1984), but topography at scales larger than 10 m may contribute as much as 20% to $\theta$ (Helfenstein 1986). Therefore, the $\theta$ values listed in Table I represent upper limits for photoclinometry.

Values of $k(\alpha)$ are shown for four values of $\omega$ and two values of $g$ in Figs. 4 and 9–13. For smooth surfaces, $\omega$ and $g$ have strong effects on $k(\alpha)$. As $\theta$ increases, however, the effects of $\omega$ and $g$ progressively diminish and nearly vanish at $\theta \approx 40°$. This is fortunate, because most photoclinometric profiles cross areas of variable intrinsic albedo, and the need to use a variable photometric parameter would make the task much more complicated. The methods of Davis and Soderblom (1984) and McEwen (1985) account for variable intrinsic albedo as opposed to brightness variations due to the topography, but assume a constant photometric behavior. Because most planetary surfaces have $\theta$ from 20° to 30°, the assumption of constant photometric behavior will usually be valid, especially if the intrinsic albedo does not vary drastically.

3.5 Fits with Lunar–Lambert Function

Unlike the Minnaert function, the lunar–Lambert function (Eqs. (4) and (5)) exhibits little or no discrepancy as a function of position on the disk. The variation of $L$ with $\alpha$ for different values of $\omega$ and for $\theta$ equal to 0° is shown in Fig. 14. For low values of $\omega$ and no macroscopic roughness, the lunar–Lambert function matches Hapke’s function nearly exactly (cf. Figs. 2, 5). With high values of $\omega$, the fit is also very good, except near the limb (Fig. 15).

Plots of $L(\alpha)$ with various values for $\theta$ and $g$ are shown in Figs. 16–20. Like Figs. 4 and 9–13 for the Minnaert function, these and Fig. 14 are the basic reference plots for use in photoclinometry. As with $k$ in the Minnaert function, $L$ becomes insensitive to $\omega$ and $g$ when $\theta$ is from 30° to 40°.

Both of the simple empirical functions considered in this paper are adequate for photoclinometry under most circumstances. However, the lunar–Lambert function gives a more accurate fit to Hapke’s function than does Minnaert’s function when viewing angles are high and when $\theta$ is less than 30° (Table I). When $\theta$ is 30° or higher, the Minnaert function shows slightly smaller values of $D_{rms}$ (Table I), but these fits do not include values near the limb, where the Minnaert function fits are poor. Therefore, the lunar–Lambert function is superior to Minnaert’s function if Hapke’s function provides an accurate description of the limb darkening of real planetary surfaces.

4. SUMMARY AND CONCLUSIONS

This paper is oriented toward providing useful photometric information for practical planetary applications, especially photoclinometry. To this end, the significant results are presented in Figs. 4, 9–13, 14, 16–20, and Table I. Tables listing the $k(\alpha)$, $L(\alpha)$, and $I(\mu, \mu_0, \alpha)$ values corresponding to each of the Hapke function parameters listed in Table I are also available from the author. The major conclusions and generalizations resulting from this effort are as follows:

1. The Minnaert and lunar–Lambert functions provide adequate matches to the $\epsilon$, $\epsilon$-dependence of Hapke’s function for most applications. These functions have the advantage of simplicity and they may be used in two-profile photoclinometry in order to separate brightness variations intrinsic to the surface materials from brightness variations due to the topography.

2. The lunar–Lambert function is closer to Hapke’s than is Minnaert’s at high viewing angles and when the macroscopic roughness is relatively low ($\theta < 30°$).
3. The dependence of the limb-darkening behavior on \( w \) and \( g \) is strong for smooth surfaces, such as laboratory samples, but greatly diminishes with increasing \( \theta \). There is almost no dependence on \( w \) or \( g \) at \( \theta = 40^\circ \). This result is fortunate for photoclinometry because the scattering function need not vary with changes in intrinsic albedo along a profile.

4. The difference in \( k \) values determined along the photometric equator and the mirror meridian at high phase angles (Goguen 1981) is small when \( \theta \) is greater than about 20°.

5. The Minnaert or lunar–Lambert fits to published Hapke models will give photoclinometric solutions that are very similar (< 1° discrepancy) to the Hapke-function solutions for nearly all of the bodies and terrains thus far modeled by Hapke’s function.

ACKNOWLEDGMENTS

Thanks go to P. Helfenstein, J. Goguen, P. Davis, R. Kirk, and R. Wildey for reviews and discussions. This research was supported by NASA Contract W-13,709 to USGS.

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